Epsilon-Delta Limit Proofs
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1 Overview
This article was designed to explain what exactly an epsilon-delta proof is and to demonstrate an example of such a proof. Although these proofs are of little practical use, they still offer key insight into what a limit actually is. In addition, they often require mathematical creativity and innovation, and developing the skills necessary for these proofs will no doubt help students solve more complicated problems. While epsilon-delta proofs are often misunderstood, this paper will attempt to explain the real mathematical principles and reasoning at work behind these problems.

2 Epsilon-Delta Proofs
What exactly is a limit? Simplistically, we can define a limit of a function as the value that function attains as that function’s variable (usually $x$) approaches a certain number. If we consider the following example

$$\lim_{x \to a} f(x) = L$$

we can determine that as $x$ approaches $a$, $f(x)$ will begin to converge to its limit, $L$. The closer $x$ gets to $a$, the closer $f(x)$ will be to $L$. We can assign more technical terms to these ideas and define a certain quantity $\delta$ as being the distance between $x$ and $a$, and we can allow $\epsilon$ to represent the distance between $f(x)$ and $L$. Graphically, $\delta$ and $\epsilon$ can be visualized as in the following figure.

Figure 1: (a) Epsilon-Delta limit of a function
We can see that as $\delta$ fluctuates, $\epsilon$ also changes. After all, the closeness of $f(x)$ to $L$ directly depends on how close $x$ is to $a$. As we narrow our distance from $a$ and allow $\delta$ to decrease, our function will get closer and closer to its limit. We can say that $a$ the limit of a certain function exists if we can satisfy the following inequalities

\begin{align}
|x - a| < \delta \\
|f(x) - L| < \delta
\end{align}

(1)

Now as long these requirements are satisfied, we can say that we have successfully proved the limit of our function. However, to satisfy these inequalities, we must do quite a bit of algebraic manipulation. From above, we can reason that $\epsilon$ is dependent on $\delta$, and therefore the two values share a certain relationship. The real crux of these proofs is determining the relationship between the inequality expressions for both $\delta$ and $\epsilon$. After we have established this relationship, we can then establish another relationship between $\epsilon$ and $\delta$ that will ensure that the above inequalities are always true. Basically, we will be able to solve for a certain $\delta$ so that the $\epsilon$ inequality is always true, thus proving our limit. Once again, to conceptualize this information, you must realize that $\epsilon$ is dependent on $\delta$, and if we can find the appropriate value for $\delta$, we can ensure that our limit exists.

On a side note, in these proofs $\delta$ and $\epsilon$ are ours to choose, and there is a large degree of mathematical freedom in determining their values. When doing these problems, you must feel comfortable with making assumptions about these quantities, and mostly all assumptions will be valid as long as they are stated correctly.

Now, we will go over a simple example of these proofs. Let us prove

$$\lim_{x \to 1} 3x - 1 = 2$$

First, we will identify what each part of the above expression stands for. 1 is our $a$, $3x - 1$ is our $f(x)$, and 2 is our $L$. Setting up our inequalities, we see

\begin{align}
|x - 1| < \delta \\
|(3x - 1) - 2| < \epsilon
\end{align}

(2)

Simplifying our $\epsilon$ inequality, we see

$$|3x - 3| < \epsilon$$

Now that we have our expressions set up, we must now set up a relationship between our $\delta$ and $\epsilon$ inequalities. Simplifying our $\epsilon$ inequality again,

$$|3(x - 1)| < \epsilon$$

$$3|x - 1| < \epsilon$$
Now, we can clearly see the relationship between our $\delta$ and $\epsilon$ expressions and we can start to develop a relationship between them

$$3|x - 1| < 3\delta = \epsilon$$

Now, we are choosing our $\epsilon$ to be equal to $3\delta$, for this ensures that our inequalities will hold. To find the correct value for $\delta$, we will simply solve for $\delta$ and find

$$\delta = \frac{\epsilon}{3}$$

Now to finalize our proof, we will substitute our values into our expressions and confirm our results

$$|x - a| = |x - 1| < \delta$$

$$|f(x) - L| = 3|x - 1| < 3\delta = 3 \left( \frac{\epsilon}{3} \right) = \epsilon$$

So our initial inequalities have been satisfied, and we have successfully proven that

$$\lim_{x \to 1} 3x - 1 = 2$$

This example was intentionally simple because I wanted to clearly explain the concepts behind epsilon-delta proofs. While you may encounter much complicated problems, if you utilize the same concepts and reasoning mentioned above, you will be able to successfully prove your limits. Above all, remember that these proofs can be solved in many ways, and being creative with your assumptions and algebraic manipulations will make these proofs much easier.